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Line defects of a two-component vector order parameter

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The line density of line defects in terms of a two-component vector order parameter are obtained from the definition of topological charges of line defects. The spatial structure and bifurcation of line defects in three-dimensional space are also studied from the topological properties of the two-component vector order parameter. The branch conditions for generating, annihilating, colliding, splitting, and merging of line defects are obtained according to the properties of the two-component vector order parameter itself. It is found that the velocities of line defects are infinite when they are being annihilated or generated, which is obtained only from the topological properties of the two-component vector order parameter. [S1063-651X(99)01709-2]

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I. INTRODUCTION

Topological defects play an important role in understanding a variety of problems in physics [1,2]. In particular, there has been progress in the study of defects associated with an n-component vector order parameter field $\vec{\phi}(\vec{r},t)$ [3–5]. For the scalar case, $n=1$, the defects are domain walls which are points for the spatial dimensionality $d=1$, lines for $d=2$, planes for $d=3$, etc. More generally, for $n=d$, one has point defects; for $n=d-1$, one generates line defects. In addition to their importance in condensed matter, these systems are also relevant to problems in cosmological structure formation. In studying these problems, questions arise as to how one can define quantities like the densities of defect and an associated defect velocity field.

It is interesting to consider an appropriate form for defect densities when expressed in terms of the vector order parameter field $\vec{\phi}(\vec{r},t)$. This has been carried out by Halperin [6], and exploited by Liu and Mazenko [7]: In the case $n=d$, the first ingredient is the rather obvious result

$$\sum_{\alpha}\delta(\vec{r}-\vec{r}_{\alpha}(t))=\delta(\vec{\phi}(\vec{r},t))\left|D\left(\frac{\phi}{x}\right)\right|,$$

where the second factor on the right-hand side is just the Jacobian of the transformation from the variable $\vec{\phi}$ to $\vec{r}$. This is combined with the less obvious result

$$\eta_{\alpha}=\operatorname{sgn}D(\phi/x)|_{\vec{r}_{\alpha}}$$

to give

$$\rho(\vec{r},t)=\sum_{\alpha}\eta_{\alpha}\delta(\vec{r}-\vec{r}_{\alpha}(t))=\delta(\vec{\phi})D(\phi/x). \tag{1}$$

In recent work [8], we showed that this analysis of Eq. (1) is incomplete, and obtained the densities of point defects directly from the definition of topological charges, and discussed what will happen when $D(\phi/x)=0$, i.e., η_l is indefinite.

For the topological line density of line defects for the case $n=d-1$,

$$\rho^{i}(\vec{r},t)=\sum_{\alpha}\int ds\frac{dr_{\alpha}^{i}}{ds}\delta(\vec{r}-\vec{r}_{\alpha}(s,t));$$

in the similar way of obtaining Eq. (1), the authors of Refs. [6,9] gave

$$\rho^{i}(\vec{r},t)=\delta(\vec{\phi})D^{i}(\phi/x), \tag{2}$$

where

$$D^{i}(\phi/x)=\epsilon^{ii_1i_2\cdots i_n}\partial_{i_1}\phi^{1}\partial_{i_2}\phi^{2}\cdots\partial_{i_n}\phi^{n}.$$

In this paper, we will investigate the line density of line defects of a two-dimensional vector order parameter, and give a complete topological analysis of the line density. This paper is organized as follows: In Sec. II, from the definition of the topological charge of line defect, the line density of line defects in terms of the two-component vector order parameter are given by means of the ϕ-mapping topological current theory [10,8]. The topological bifurcation of line defects in three-dimensional space is also given. In Sec. III, from the topological properties of the two-component vector order parameter, the conditions for generating, annihilating, colliding, splitting, and merging line defects are obtained, and several crucial cases of branch process are discussed in detail. We present our concluding remarks in Sec. IV.

II. SPATIAL STRUCTURE OF LINE DEFECTS

A. Line density of line defects

Let us study a two-component vector order parameter $\vec{\phi}(\vec{r},t)$ at a fixed time, which is denoted as $\vec{\phi}(\vec{r})$, and take a cross section normal to the z axis with coordinates $x^1=x$ and $x^2=y$; the intersection points between the line defects and the cross section are just the zero points of the two-component vector order parameter $\vec{\phi}$, i.e.,

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$$\phi^1(x,y)=0,$$
$$\phi^2(x,y)=0. \tag{3}$$

If the Jacobian determinant

$$D(\phi/x)=\tfrac{1}{2}\epsilon^{jk}\epsilon_{ab}\partial_j\phi^a\partial_k\phi^b\neq 0, \quad j,k=1,2, \tag{4}$$

the solutions of Eqs. (3) are generally expressed as

$$x=x_l, \quad y=y_l, \quad l=1,2,\ldots,N, \tag{5}$$

which represent N zero points $\vec{x}_l=(x_l,y_l)$ on this cross section. ϵ^{jk} and ϵ_{ab} are fully antisymmetric tensors, and the summation is over repeated indices in Eq. (4).

The topological charge of the lth line defect [or the generalized winding number W_l of $\vec{\phi}$ at one of zero points (x_l,y_l)] is defined by the Gauss map n: $\partial\Sigma_l\rightarrow S^1$ [10],

$$W_l=\frac{1}{2\pi}\int_{\partial\Sigma_l}n^*(\epsilon_{ab}n^a dn^b), \quad n^a=\phi^a/\|\phi\|, \tag{6}$$

where n^* is the pullback of the Gauss map n, and $\partial\Sigma_l$ is the boundary of a neighborhood Σ_l of $\vec{x}_l$. $\Sigma_l\cap\Sigma_m=\emptyset$ for Σ_m is the neighborhood of another arbitrary zero point $\vec{x}_m$. In topology this means that, when the point $\vec{x}$ covers $\partial\Sigma_l$ once, the unit vector $\vec{n}$ will cover S^1, or $\vec{\phi}$ covers the corresponding region W_l times, which is a topological invariant. Using the Stokes' theorem in the exterior differential form, one can deduce that

$$W_l=\frac{1}{2\pi}\int_{\Sigma_l}\epsilon_{ab}\epsilon^{jk}\partial_j n^a\partial_k n^b d^2x. \tag{7}$$

So it is clear that the topological charge density of line defects (or the topological charges densities) on the cross section is just

$$\rho_z=\frac{1}{2\pi}\epsilon_{ab}\epsilon^{jk}\partial_j n^a\partial_k n^b, \quad j,k=1,2, \tag{8}$$

Similarly, we may obtain the topological charge density line defects on a cross section normal to the y axis,

$$\rho_y=\frac{1}{2\pi}\epsilon_{ab}\epsilon^{jk}\partial_j n^a\partial_k n^b, \quad j,k=1,3, \tag{9}$$

and the topological charge density of line defects on a cross section normal to the x axis:

$$\rho_x=\frac{1}{2\pi}\epsilon_{ab}\epsilon^{jk}\partial_j n^a\partial_k n^b, \quad j,k=2,3. \tag{10}$$

The line density of line defects normal to one plane give the topological charge density of line defects on the plane [6], and one can construct *the line density of line defects* in three-dimensional space according to Eqs. (8), (9) and (10):

$$\rho^i=\tfrac{1}{2}\epsilon^{ijk}\epsilon_{ab}\partial_j n^a\partial_k n^b, \quad i,j,k=1,2,3, \tag{11}$$

where

$$\rho^1=\rho_x, \quad \rho^2=\rho_y, \quad \rho^3=\rho_z.$$

Because ϵ^{ijk} is a fully antisymmetric tensor, it is easy to see that the divergence of the line density of the line defects is zero,

$$\vec{\nabla}\cdot\vec{\rho}=\partial_i\rho^i=0, \tag{12}$$

which is the reason that line defects occur on a set of one-dimensional curves that may be either closed loops or infinite curves. Using the same methods in as Ref. [11], one can obtain that

$$\rho^i=\delta^2(\vec{\phi})D^i\left(\frac{\phi}{x}\right), \tag{13}$$

where

$$D^i\left(\frac{\phi}{x}\right)=\frac{1}{2}\epsilon^{ijk}\epsilon_{ab}\partial_j\phi^a\partial_k\phi^b, \quad i,j,k=1,2,3.$$

Here one can see that the line density of line defects in terms of the two-component vector order parameter (13) is obtained directly from the definition of topological charge of the line defect (the winding number of zero points), which is useful because it avoids the problem of having to specify the position of line defects explicitly, and is more general than usually considered. From Eq. (13) we see that $\vec{\rho}$ does not vanish only at the zero points of $\vec{\phi}$ in three-dimensional space, i.e.,

$$\phi^1(x,y,z)=0, \quad \phi^2(x,y,z)=0. \tag{14}$$

When

$$\vec{D}\left(\frac{\phi}{x}\right)=\left[D^1\left(\frac{\phi}{x}\right),D^2\left(\frac{\phi}{x}\right),D^3\left(\frac{\phi}{x}\right)\right]\neq 0,$$

the solutions of Eqs. (14) are

$$x=x_l(s), \quad y=y_l(s), \quad z=z_l(s), \quad l=1,2,\ldots,N, \tag{15}$$

which represent N line defects $L_l(l=1,2,\ldots,N)$ where $\vec{\phi}(\vec{r})=\vec{0}$ in three-dimensional space. The direction of the lth line defect is determined by $\vec{D}(\phi/x)$ on L_l [11].

In the theory of the δ function of $\vec{\phi}(\vec{r})$, one can prove that [11]

$$\vec{\rho}=\sum_{l=1}^{N}\beta_l\eta_l\int_{L_l}ds\frac{d\vec{r}_l}{ds}d\vec{r}_l\delta^3(\vec{r}-\vec{r}_l), \tag{16}$$

where the positive integer β_l is called the Hopf index, and $\eta_l=\pm 1$ is the Brouwer degree of map $x\rightarrow\phi$ [11]. One can find a relation between the Hopf index β_l, the Brouwer degree η_l, and the winding number W_l: $W_l=\beta_l\eta_l$ [8]. Let Σ be an arbitrary surface, and suppose that M line defects pass through it. According to Eq. (13), one can prove that

$$\int_{\Sigma}\vec{\rho}\cdot d\vec{\sigma}=\sum_{l=1}^{M}\beta_l\eta_l, \tag{17}$$

which confirms that $\vec{\rho}$ represents the line density of line defects in space.

Here we see that result (2) obtained by Halperin and Mazenko and co-workers is not complete. They only considered the case $\beta_l=1$, and did not discuss what will happen when $\vec{D}(\phi/x)=0$, i.e., η_l is indefinite, which we will discuss in Sec. II B.

B. Spatial bifurcation of line defects

Solution (15) of Eqs. (14) is based on the condition that the Jacobian $\vec{D}(\phi/x)\neq\vec{0}$. When the condition fails, the above results (15) will change in some way. It is interesting to discuss what will happen, and what the correspondence in physics will be when

$$\vec{D}\left(\frac{\phi}{x}\right)=\vec{0} \tag{18}$$

at some points $\vec{r}_l^*$ along L_l. This restrictive condition will lead to an important fact that the functional relationship between z and x, or z and y, is not unique in the neighborhood of $\vec{r}_l^*$. This fact is easily seen from

$$\frac{dx}{dz}=\left.\frac{D^1\left(\frac{\phi}{x}\right)}{D^3\left(\frac{\phi}{x}\right)}\right|_{\vec{r}_l^*},\quad \frac{dy}{dz}=\left.\frac{D^2\left(\frac{\phi}{x}\right)}{D^3\left(\frac{\phi}{x}\right)}\right|_{\vec{r}_l^*}, \tag{19}$$

which under Eq. (18) directly shows that the direction of the integral curve of Eq. (19) is indefinite at $\vec{r}_l^*$. Therefore, the very point $\vec{r}_l^*$ is called a bifurcation point of the two-dimensional vector order parameter in three-dimensional space.

According to the ϕ-mapping topological current theory, the Taylor expansion of the solution of Eqs. (14) in the neighborhood of the bifurcation point $\vec{r}_l^*$ can be generally expressed as [10]

$$A(x-x_l^*)^2+2B(x-x_l^*)(z-z_l^*)+C(z-z_l^*)^2+\cdots=0,$$

which leads to

$$A\left(\frac{dx}{dz}\right)^2+2B\frac{dx}{dz}+C=0 \tag{20}$$

and

$$C\left(\frac{dz}{dx}\right)^2+2B\frac{dz}{dx}+A=0. \tag{21}$$

The solutions of Eqs. (20) or (21) give different directions of the branch curves (zero lines of the two-component vector order parameter, i.e., line defects) at the bifurcation point. There are four important cases.

Case (1) (A≠0): For $\Delta=4(B^2-AC)>0$, from Eq. (20), we obtain two different directions of the line defects in three-dimensional space,

$$\left.\frac{dx}{dz}\right|_{1,2}=\frac{-B\pm\sqrt{B^2-AC}}{A}, \tag{22}$$

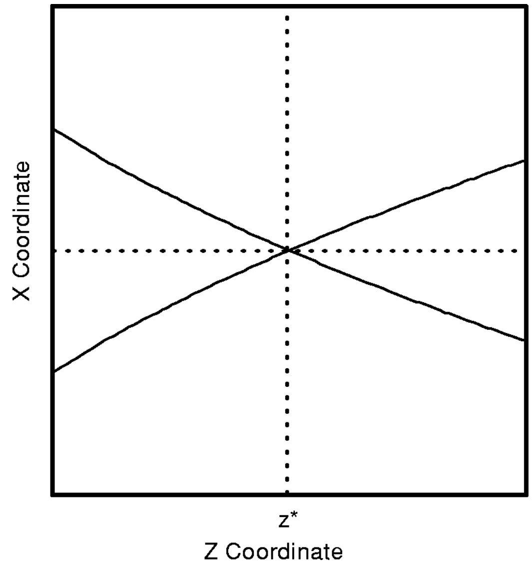


FIG. 1. Bifurcation solution for Eq. (22): two line defects intersect at the bifurcation point in three-dimensional space.

which is shown in Fig. 1, where two zero lines intersect with different directions at $\vec{r}_l^*$. This shows that two line defects intersect at the bifurcation point.

Case (2) (A≠0): For $\Delta=4(B^2-AC)=0$, from Eq. (20), we obtain only one direction of the line defects in three-dimensional space,

$$\left.\frac{dx}{dz}\right|_{1,2}=-\frac{B}{A}, \tag{23}$$

which includes three important cases according to Fig. 2. First, two zero lines tangentially contact, i.e., two line defects tangentially intersect at the bifurcation point [see Fig. 2(a)]. Second, two zero lines merge into one zero line, i.e., two line defects merge into one line defect at the bifurcation point [see Fig. 2(b)]. Finally, one zero line resolves into two, i.e., one line defect splits into two at the bifurcation point [see Fig. 2(c)].

Case (3) (A=0,C≠0): For $\Delta=4(B^2-AC)\neq0$, from Eq. (21), we have

$$\left.\frac{dz}{dx}\right|_{1,2}=\frac{-B\pm\sqrt{B^2-AC}}{C}=0,-\frac{2B}{C}. \tag{24}$$

As shown in Fig. 3, there are two important cases: (a) One zero line resolves into three, i.e., one line defect splits into three line defects at the bifurcation point [see Fig. 3(a)]. (b) Three zero lines merge into one zero line, i.e., three line defects merge into one at the bifurcation point [see Fig. 3(b)].

Case (4) (A=C=0): Equations (20) and (21), respectively, give

$$\frac{dx}{dz}=0,\quad \frac{dz}{dx}=0. \tag{25}$$

This case is obvious as in Fig. 4, which is similar to case 3.

The above solutions reveal the spatial bifurcation structure of line defects in three-dimensional space. In addition to the intersection of line defects, i.e., two line defects intersect

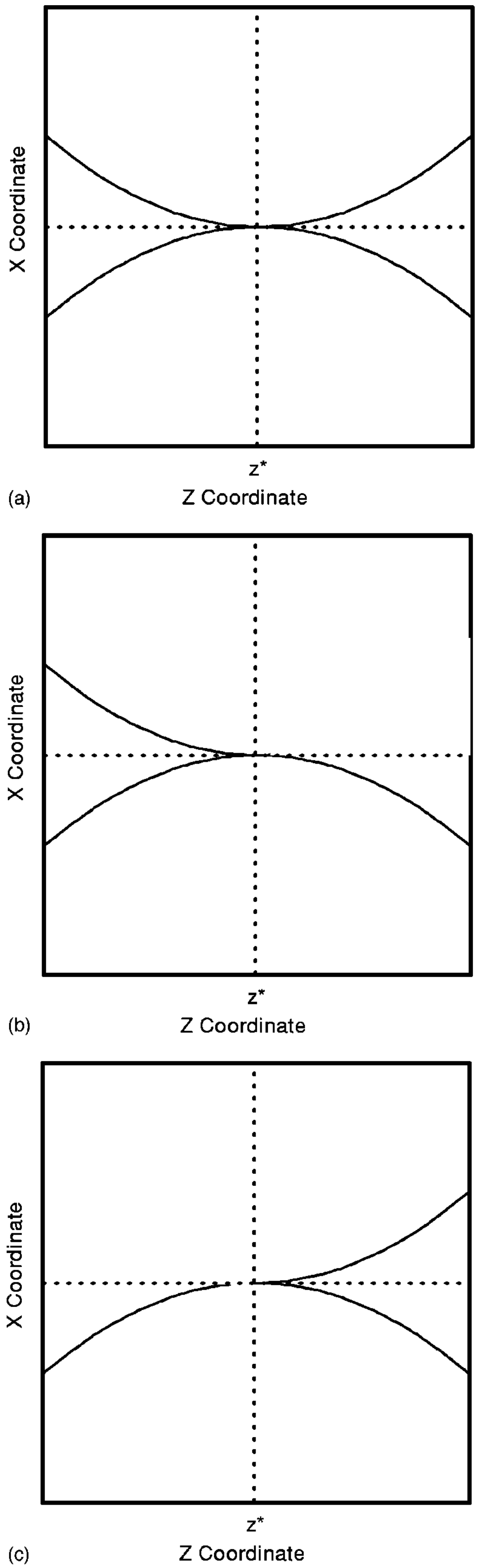


FIG. 2. Bifurcation solutions for Eq. (23): line defects have the same direction of tangent when they intersect in three-dimensional space. (a) Two line defects tangentially contact at the bifurcation point. (b) Two line defects merge into one line defect at the bifurcation point. (c) One line defect splits into two line defects at the bifurcation point.

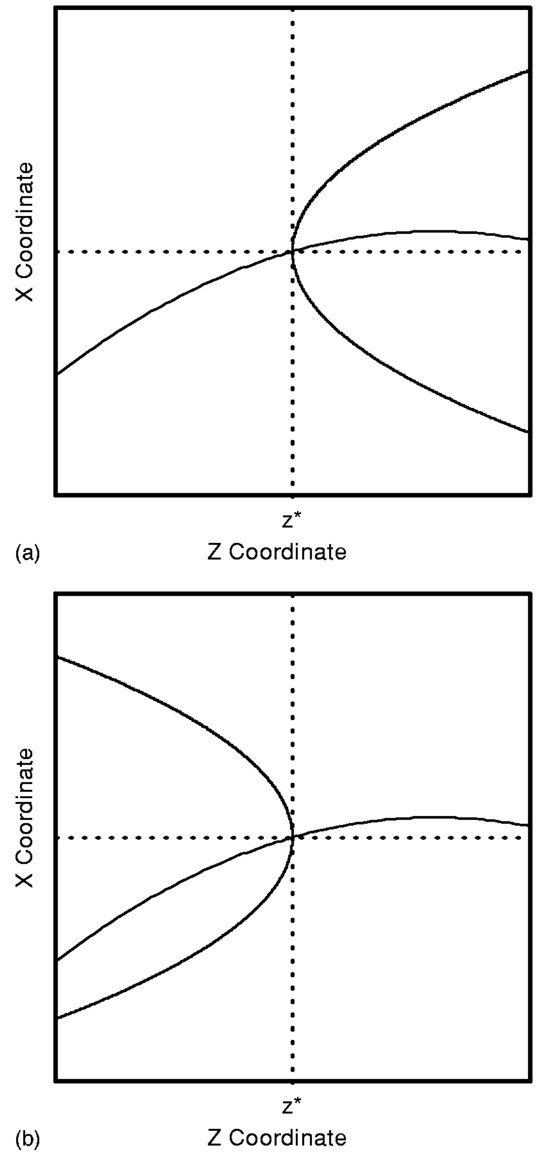


FIG. 3. Two important cases of Eq. (24). (a) One line defects splits into three line defects at the bifurcation point in three-dimensional space. (b) Three line defects merge into one line defect at the bifurcation point.

at the bifurcation point [see Figs. 1 and 2(a)], splitting and merging of line defects are also included. When a multi-charged line defect passes through the bifurcation point in three-dimensional space, it may split into several line defects along different branch curves [see Figs. 2(c), 3(a), and 4(b)]; moreover, several line defects can merge into one line defect at the bifurcation point [see Figs. 2(b), 3(b), and 4(a)]. For the divergence of the line density of line defects to be zero [Eq. (12)], the sum of the topological charges of the final line defect(s) must be equal to that of the initial line defect(s) at the bifurcation point, i.e.,

$$\sum_f \beta_{l_f}\eta_{l_f}=\sum_i \beta_{l_i}\eta_{l_i} \tag{26}$$

for fixed l.

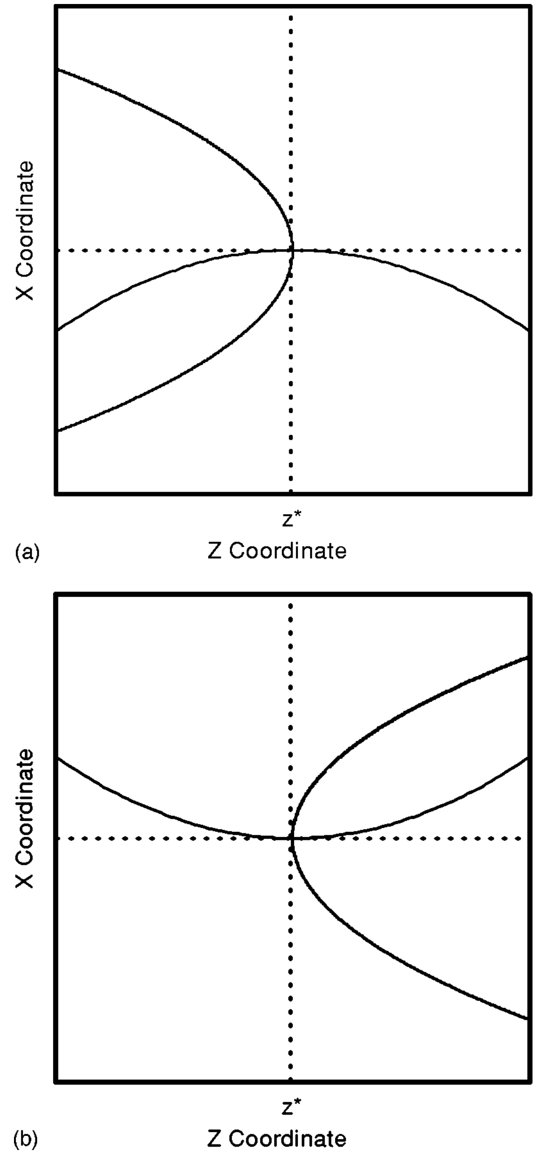


FIG. 4. (a) Three line defects merge into one at the bifurcation point. (b) One line defect splits into three line defects at the bifurcation point.

III. EVOLUTION OF LINE DEFECTS

In Sec. II, we did not consider the motion of line defects, and only discussed the space structure of line defects in three-dimensional space. In this section, we will investigate the evolution of a line defect in $(3+1)$-dimensional space-time with coordinates $x^1=x$, $x^2=y$, $x^3=z$, and $x^0=t$.

A. Continuity equation of line density of line defects

From Eq. (11), we construct a topological tensor current of the two-component vector order parameter

$$K^{\mu\nu}=\frac{1}{2\pi}\epsilon^{\mu\nu\lambda\sigma}\epsilon_{ab}\partial_\lambda n^a\partial_\sigma n^b,\qquad \mu,\nu=0,1,2,3,$$

where

$$\rho^i=K^{0i}=\frac{1}{2\pi}\epsilon^{0ijk}\epsilon_{ab}\partial_j n^a\partial_k n^b.$$

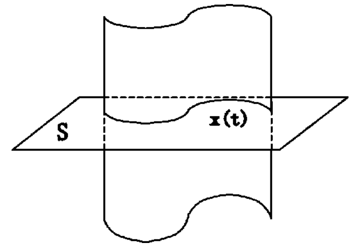


FIG. 5. S is one cross section normal to the z axis. $X(t)$ is the intersection line between the evolution surface of a line defect and the cross section S, i.e., the movement curve of the line defect on the cross section S.

Following the ϕ-mapping topological current theory, it can be proved that

$$K^{\mu\nu}=\delta^2(\vec{\phi})D^{\mu\nu}\left(\frac{\phi}{x}\right),$$

where

$$D^{\mu\nu}(\phi/x)=\tfrac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\epsilon_{ab}\partial_\lambda\phi^a\partial_\sigma\phi^b.$$

Considering that $\epsilon^{\mu\nu\lambda\sigma}$ is a fully antisymmetric tensor, we can prove that

$$\partial_\mu K^{\mu\nu}=0;$$

that is,

$$\partial_t\rho^i+\partial_j K^{ji}=0, \tag{27}$$

which is just the continuity equation satisfied by ρ^i [9].

B. Motion of line defects on a cross section

When we investigate the movement of line defects, there exist evolution surfaces formed by the movements of the line defects in $(3+1)$-dimensional space-time. For simplicity, let us take an arbitrary cross section normal to the z axis, i.e., $(2+1)$-dimensional space-time with coordinates $x^1=x$, $x^2=y$, and $x^0=t$. The intersection lines between the evolution surfaces and the cross section are just motion curves of line defects on the cross section (see Fig. 5). One thing to point out is that if one takes the cross sections all along the axis, the motion properties of the line defects will be given completely.

From the continuity equation satisfied by ρ^i [Eq. (27)], we can give the continuity equation of the line densities of line defects on this cross section:

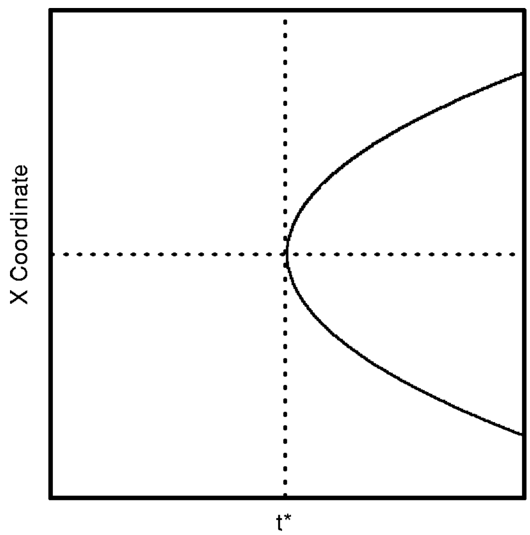
$$\partial_t\rho_z+\partial_j K^j=0,\qquad j=1,2, \tag{28}$$

where

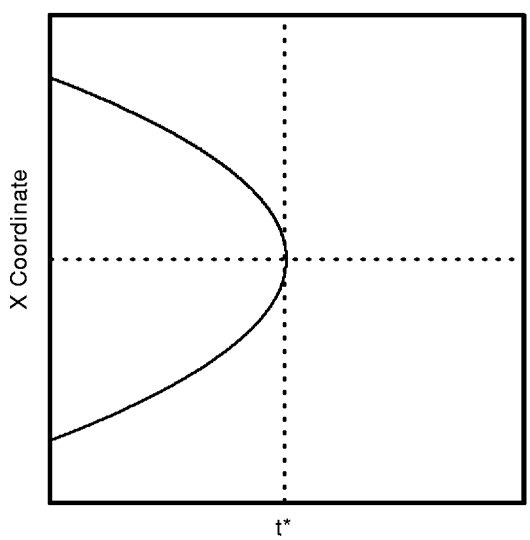
$$\rho_z=K^{03}=\delta^2(\vec{\phi})D^0\left(\frac{\phi}{x}\right) \tag{29}$$

and

$$K^j=K^{j3}=\delta^2(\vec{\phi})D^j\left(\frac{\phi}{x}\right),\qquad j=1,2, \tag{30}$$



(a)



(b)

FIG. 6. (a) The origin of line defects, i.e., two line defects are generated at the limit point. (b) Two line defects are annihilated at the limit point.

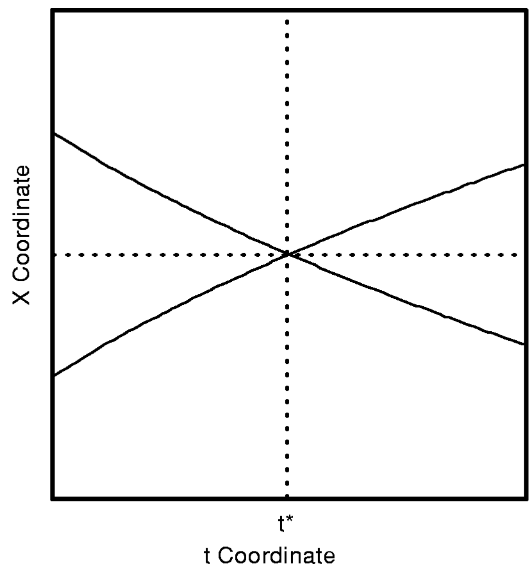


FIG. 7. Two line defects collide with different directions of motion at the bifurcation point in $(3+1)$-dimensional space-time.

where the Jacobians $D^{\mu}(\phi/x)=\frac{1}{2}\epsilon^{\mu\nu\lambda}\epsilon_{ab}\partial_{\nu}\phi^{a}\partial_{\lambda}\phi^{b}$ $(\mu=0,1,2,3)$.

From Eq. (29) the line densities of line defects on the cross section do not vanish only at the zero points of the vector order parameter $\vec{\phi}(x,y,t)$, i.e.,

$$\phi^{1}(x,y,t)=0, \quad \phi^{2}(x,y,t)=0, \tag{31}$$

which determine the positions of line defects. If the Jacobian determinant $D^{0}(\phi/x)\neq 0$, the solutions of Eqs. (31) are expressed as

$$x=x_{l}(t), \quad y=y_{l}(t), \quad l=1,2,\ldots,N, \tag{32}$$

which represent the motion curves of the N zero point $\vec{x}_{l}(t)$ on the cross section, and which show them moving in $(2+1)$-dimensional space-time.

According to Eq. (16), we obtain that

$$\rho_{z}(x,y,t)=\sum_{l=1}^{N}\beta_{l}\eta_{l}\delta^{2}(\vec{x}-\vec{x}_{l}(t)). \tag{33}$$

Following our theory, we can also obtain the velocity of the lth zero point on the cross section,

$$\vec{v}_{l}=\frac{d\vec{x}_{l}}{dt}=\left.\frac{\vec{D}(\phi/x)}{D^{0}(\phi/x)}\right|_{\vec{x}_{l}}, \quad \vec{D}(\phi/x)=(D^{1}(\phi/x),D^{2}(\phi/x)), \tag{34}$$

from which one can identify the zero-point velocity field on the cross section,

$$\vec{v}(x,y,t)=\frac{\vec{D}(\phi/x)}{D^{0}(\phi/x)}, \tag{35}$$

where it is assumed that the velocity field is used inside expressions multiplied by the zero points, locating the δ function. The expressions given by Eq. (35) for the velocity field of the zero points are useful because they avoid the problem of having to specify the positions of the zero points explicitly. The positions are implicitly determined by the zeros of the two-component vector order parameter $\vec{\phi}$ on the cross section. So the location and the velocity of the lth zero point are determined by the lth zero $\vec{x}_{l}(t)$ and the vector field $\vec{v}(x,y,t)$ on $\vec{x}_{l}(t)$, respectively.

The current densities of the line defects on the cross section can be written in the same form as the current densities in hydrodynamics:

$$J^{i}=\sum_{l=1}^{N}\beta_{l}\eta_{l}\delta^{2}(\vec{x}-\vec{x}_{l}(t))\frac{dx_{l}^{i}}{dt}, \quad i=1,2. \tag{36}$$

From Eqs. (30), (33), and (34), the current densities can be written as the concise forms

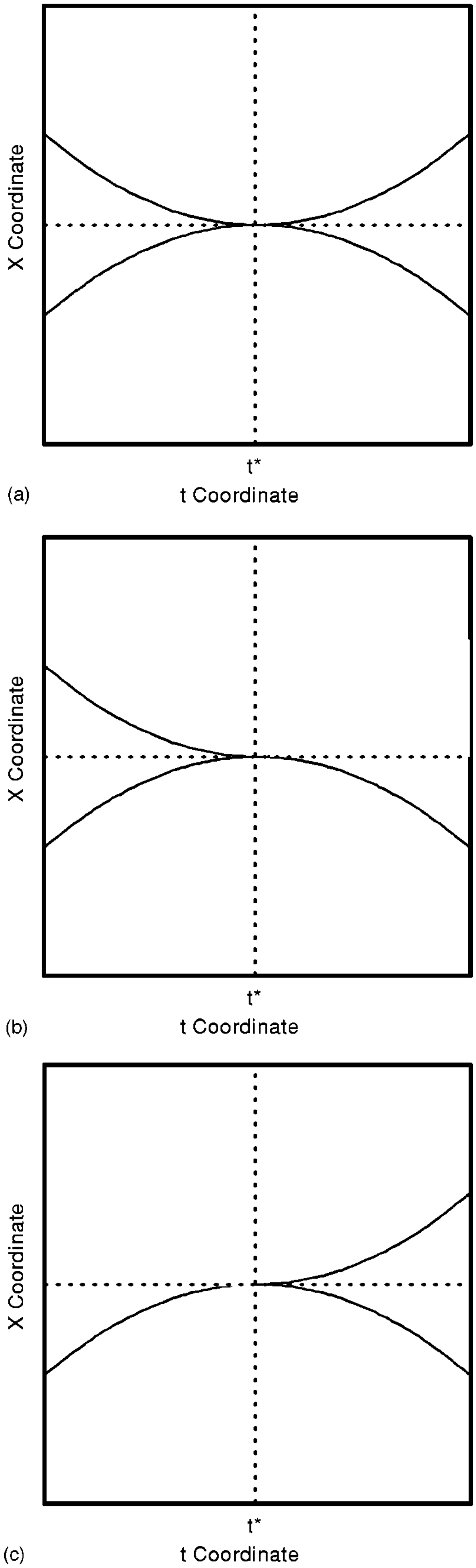


FIG. 8. Line defects have the same direction of motion in (3 +1)-dimensional space-time. (a) Two line defects collide at the bifurcation point. (b) Two line defects merge into one line defect at the bifurcation point. (c) One line defect splits into two line defects at the bifurcation point.

$$J^i = K^i = \delta^2(\vec{\phi}) D^i\left(\frac{\phi}{x}\right) = \rho_z v^i. \tag{37}$$

According to Eq. (28), the topological charges of line defects on the cross section are conserved,

$$\frac{\partial \rho_z}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0, \tag{38}$$

which is only the topological property of the vector order parameter. This is the true reason that the singularity (topological defect) with a winding number $W \neq 0$ cannot be removed without tampering with the order parameter at arbitrarily great distances from the singular point, and that singular configurations with different winding numbers cannot be transformed into one another by local surgery, as pointed out by Mermin [1].

C. Generation and annihilation of line defects

Solutions (32) of Eqs. (31) are based on the condition that the Jacobian $D^0(\phi/x) \neq 0$. It is interesting to discuss what will happen and what the correspondence in physics will be when this condition fails. When $D^0(\phi/x) = 0$, i.e., η_l is indefinite, it is shown that there exist several crucial cases of branch process. There are two kinds of branch points, namely, limit points and bifurcation points. Each of them corresponds to different cases of branch process.

First, we study the case when the zeros of the two-component vector order parameter $\vec{\phi}(x,y,t)$ include some limit points. The limit points are determined by Eqs. (31) and

$$D^0\left(\frac{\phi}{x}\right) = 0, \quad D^1\left(\frac{\phi}{x}\right) \neq 0 \tag{39}$$

or

$$D^0\left(\frac{\phi}{x}\right) = 0, \quad D^2\left(\frac{\phi}{x}\right) \neq 0. \tag{40}$$

For simplicity, we only consider Eqs. (39), and denote one of the limit points as $(\vec{x}^*, t^*)$. Taking account of Eqs. (39) and using the implicit function theorem, we have a unique solution of Eqs. (31) in the neighborhood of the limit point $(\vec{x}^*, t^*)$,

$$t = t(x), \quad y = y(x), \tag{41}$$

with $t^* = t(x^*)$ and $y^* = y(x^*)$. From Eqs. (39), it is easy to see

$$\left.\frac{dt}{dx}\right|_{(\vec{x}^*, t^*)} = 0, \quad \text{i.e., } \left.\frac{dx}{dt}\right|_{(\vec{x}^*, t^*)} = \infty. \tag{42}$$

Thus, the Taylor expansion of solution (41) in the neighborhood of the limit point $(\vec{x}^*, t^*)$ is

$$t - t^* = \frac{1}{2} \left.\frac{d^2 t}{(dx)^2}\right|_{(\vec{x}^*, t^*)} (x - x^*)^2. \tag{43}$$

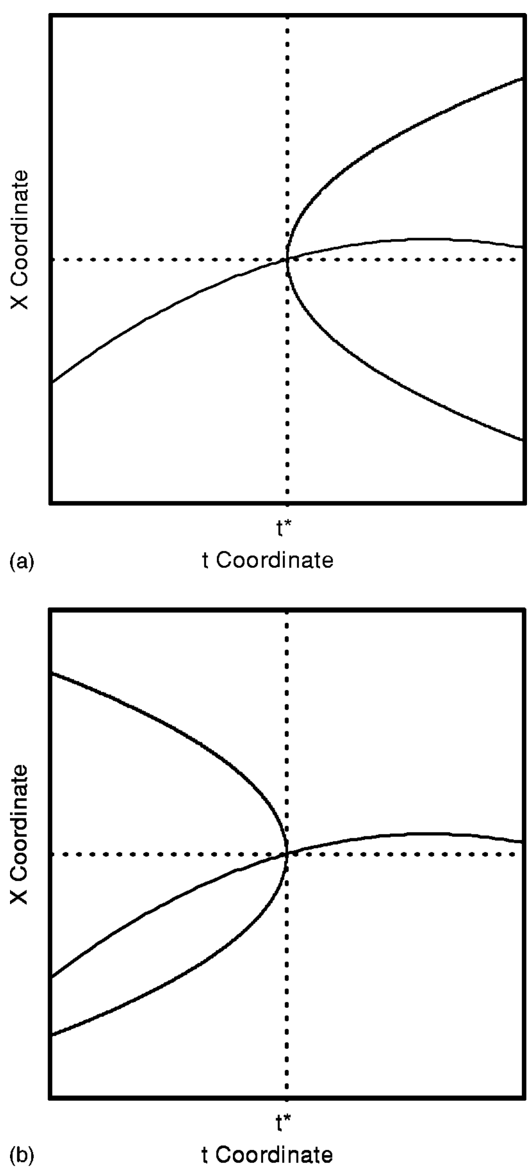


FIG. 9. (a) One line defect splits into three line defects at the bifurcation point in $(3+1)$-dimensional space-time. (b) Three line defects merge into one line defect at the bifurcation point.

From Eq. (43) we can obtain the branch solutions of zero points at the limit point. If $d^2t/(dx)^2|_{(\vec{x}^*,t^*)}>0$, we have the branch solutions for $t\geqslant t^*$ [Fig. 6(a)]; otherwise, we have the branch solutions for $t\leqslant t^*$ [Fig. 6(b)]. The former is related to the generation of the line defects, and the latter is related to the annihilation of the line defects. Since the topological charge is identically conserved, the topological charges of these two generated or annihilated line defects must be opposite, i.e., $\beta_1\eta_1+\beta_2\eta_2=0$, which shows the generation and annihilation of a line defect and antidefect pair. This can explain why a pair of defects with winding numbers W and $-W$ is equivalent to a nonsingular configuration, and why the defects can annihilate one another within a bounded region without the need for any rearrangement of the order-parameter field at large distance [1]. One of the results of Eq. (43), that the velocity of line defects is infinite when they are being annihilated, agrees with that obtained by Bray [12], who has a scaling argument associated with defect final annihilation which leads to a large velocity tail, and Mazenko [13], who claimed that there is a large velocity tail in the line defect velocity distribution corresponding to the annihilation of the defect. From Eq. (43), we also obtain the result that the velocity of the line defects is infinite when they are being generated, which is gained only from the topology of the two-component vector order parameter.

D. Collision, splitting, and mergence of line defects

Let us turn to the other case, in which the restrictions of Eqs. (31) are

$$D^j\left(\frac{\phi}{x}\right)=0, \quad j=0,1,2. \tag{44}$$

These three restrictive conditions will lead to the important fact that the functional relationship between t and x or t and y is not unique in the neighborhood of $(\vec{x}^*,t^*)$. In our topological current theory, this fact is easily seen from

$$\frac{dx}{dt}=\left.\frac{D^1(\phi/x)}{D^0(\phi/x)}\right|_{(\vec{x}^*,t^*)}, \quad \frac{dy}{dt}=\left.\frac{D^2(\phi/x)}{D^0(\phi/x)}\right|_{(\vec{x}^*,t^*)} \tag{45}$$

which under Eq. (44) directly shows that the direction of the integral curve of Eqs. (45) is indefinite at $(\vec{x}^*,t^*)$. Therefore, the very point $(\vec{x}^*,t^*)$ is called a bifurcation point of the two-component vector order parameter $\vec{\phi}(x,y,t)$.

According to the ϕ-mapping topological current theory, the Taylor expansion of the solution of Eqs. (31) in the neighborhood of the bifurcation point $(\vec{x}^*,t^*)$ can be generally expressed as [10]

$$\alpha(x-x^*)^2+2\beta(x-x^*)(t-t^*)+\gamma(t-t^*)^2+\cdots=0, \tag{46}$$

which leads to

$$\alpha\left(\frac{dx}{dt}\right)^2+2\beta\frac{dx}{dt}+\gamma=0 \tag{47}$$

and

$$\gamma\left(\frac{dt}{dx}\right)^2+2\beta\frac{dt}{dx}+\alpha=0, \tag{48}$$

where α, β, and γ are constants determined by the two-component vector order parameter $\vec{\phi}$. The solutions of Eqs. (47) or (48) give different motion directions of zero points on the cross section at the bifurcation point. There are four possible cases, which will show the physical meanings of the bifurcation points.

Case (1) ($\alpha\neq 0$): For $\Delta=4(\beta^2-\alpha\gamma)>0$, from Eq. (47) we obtain two different motion directions of the zero point on the cross section,

$$\left.\frac{dx}{dt}\right|_{1,2}=\frac{-\beta\pm\sqrt{\beta^2-\alpha\gamma}}{\alpha}, \tag{49}$$

which is shown in Fig. 7, where two worldlines of two zero points intersect with different directions at the bifurcation point on the cross section. This shows that two line defects meet and then depart at the bifurcation point.

Case (2) ($\alpha \neq 0$): For $\Delta = 4(\beta^2 - \alpha\gamma) = 0$, from Eq. (47) we obtain only one motion direction of the zero point on the cross section,

$$\left.\frac{dx}{dt}\right|_{1,2} = -\frac{\beta}{\alpha}. \tag{50}$$

which includes three important cases. (a) Two worldlines of zero points tangentially contact, i.e., two line defects collide at the bifurcation point [see Fig. 8(a)]. (b) Two worldlines of zero points merge into one worldline, i.e., two line defects merge into one line defect at the bifurcation point [see Fig. 8(b)]. (c) One worldline resolves into two worldlines, i.e., one line defect splits into two line defects at the bifurcation point [see Fig. 8(c)]. Now it is clear that a pair of defects can be transformed into a single defect with a total net winding number, without requiring surgery to extend beyond the interior of any contour surrounding the pair, considered by Mermin [1].

Case (3) ($\alpha = 0, \gamma \neq 0$): For $\Delta = 4(\beta^2 - \alpha\gamma) \neq 0$, from Eq. (48), we have

$$\left.\frac{dt}{dx}\right|_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - \alpha\gamma}}{\gamma} = 0, \quad -\frac{2\beta}{\gamma}. \tag{51}$$

There are two important cases: (a) One worldline of resolves into three worldlines, i.e., one line defect splits into three line defects at the bifurcation point [see Fig. 9(a)]. (b) Three worldlines merge into one worldline, i.e., three line defects merge into one line defect at the bifurcation point [see Fig. 9(b)].

Case (4) ($\alpha = \gamma = 0$): Equations (47) and (48), respectively, give

$$\frac{dx}{dt} = 0, \quad \frac{dt}{dx} = 0. \tag{52}$$

This case shows that two worldlines intersect normally at the bifurcation point, which is similar to case (3): (a) Three line defects merge into one line defect at the bifurcation point. (b) One line defect splits into three line defects at the bifurcation point.

Now the evolution of the line defects is investigated in detail: at the limit points of the two-component vector order parameter, line defects are generated or annihilated; at the bifurcation points of the two-component vector order parameter, line defects collide, split, or merge. The identical conversation of the topological charge shows that the sum of the topological charge of the final line defect(s) must be equal to that of the initial line defect(s) at the bifurcation point. Furthermore, from above studies, we see that the generation, annihilation, collision, splitting, and merging of line defects are not gradual changes, but start at a critical value of arguments, i.e., a sudden change.

IV. CONCLUSIONS

First, we give the spatial structure of line defects in three-dimensional space. The line densities of line defects (13) and (16) are obtained directly from the definition of topological charges of line defects, which is more general than usually considered. When $\vec{D}(\phi/x) = 0$, the intersection, splitting, and merging of line defects in three-dimensional space are investigated in detail by making using of the ϕ-mapping topological current theory. Second, the evolution of line defects in $(3+1)$-dimensional space-time is studied. There exist crucial cases of branch processes in the evolution of line defects when the Jacobian $D^0(\phi/x) = 0$, i.e., η_l is indefinite: At one of the limit points of the vector order parameter, a pair of line defects with opposite topological charge can be annihilated or generated. At one of the bifurcation points of the vector order parameter, a line defect with topological charge W may split into several line defects (total topological charges is W); conversely, several line defects (total topological charges is W) can merge into line defect with a topological charge W. Also, at one of the bifurcation points of the vector order parameter, two line defects meet and then depart. These show that line defects are unstable at these branch points of the vector order parameter. From the topological properties of the vector order parameter, we obtained that the velocity of the line defects is infinite when they are being annihilated or generated, which agrees with what was obtained by Bray [12] and Mazenko [13]. Finally, we would like to point out that all the results in this paper are obtained from only the viewpoint of topology, without using any particular models or hypotheses.

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